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DETERMINATION OF STRESSES IN ELLIPSOIDAL RIGID INCLUSIONS

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A problem of determining stresses in isolated ellipsoidal rigid inclusions contained in an isotropic elastic space exposed to the impact of external forces uniformly distributed at infinity is considered. Examples of inclusions in the form of oblate and prolate spheroids are studied when the problem has a unique solution.

Key words: *isolated rigid inclusions, uniform stress field, oblate and prolate spheroids.*

It was demonstrated [1] by considering plane problems of determining the stress–strain state of an elastic area with an arbitrarily shaped rigid inclusion that the stress field in this inclusion is determined unambiguously. In a plane with the elliptic inclusion, the stress field is uniform and the stresses at infinity and in the inclusion are bound with one-to-one relations.

The possibility of determining the stresses in the ellipsoidal rigid inclusion (ERI) of the general type cannot be proved. Nevertheless, it turned out to be possible for ERIs shaped as oblate and prolate spheroids, which are considered in the present paper.

1. Elastic Domain with Isolated ERIs. We consider an elastic domain with ERIs exposed to the impact of stresses uniformly distributed at infinity σ_{kl}^{∞} $(k,l=1,2,3)$. The distance between the centers of two arbitrary inclusions is large, as compared to their sizes; therefore, the impact of one of them on the stress state of any other inclusion can be ignored (isolated ERI). Under these assumptions, the problem solution reduces to independent solving of N (N is the total number of all ERIs) problems on determining the stress–strain state of the elastic domain v with one inclusion v^* under the impact of external stresses at infinity. A similar problem was studied in [2] for a physically nonlinear ellipsoidal inclusion (PNEI) with constitutive equations of a rather general form $\varepsilon^* = F(\sigma^*)$ and $\sigma^* = G(\varepsilon^*)$ (F and G are nonlinear tensor operators acting on the stress tensor σ^* and strain tensor ε^*). In the domain v, Hooke's law is valid $\varepsilon = a : \sigma$ and $\sigma = b : \varepsilon$ (a and b are the reciprocal tensors of elastic compliances and elastic moduli, respectively).

The following relations between the stress–strain state in the PNEI and at infinity were established in [2]:

$$\varepsilon^* = \varepsilon^{\infty} + S : (\varepsilon^* - \tilde{\varepsilon}^*), \qquad \varepsilon^{\infty} = a : \sigma^{\infty}, \qquad \tilde{\varepsilon}^* \equiv a : \sigma^*. \tag{1.1}$$

Here, the fourth rank tensor S is independent of the coordinates x_k $(k = 1, 2, 3)$ and is determined by the geometry of the domain v^* and by the elastic characteristics of the ambient medium v. For the isotropic domain v, the components of the tensor S are given in [2].

For the rigid inclusion v^* considered, Eq. (1.1) at $\varepsilon^* = 0$ yields

$$S : \tilde{\varepsilon}^* = \varepsilon^{\infty}. \tag{1.2}$$

It can be inferred from Eqs. (1.1) and (1.2) that the components of the stress tensor σ^* as functions of the tensor ε^{∞} specified at infinity can be found if the determinant of the 6×6 matrix corresponding to the tensor S [2] differs from zero. Note that the stress field in the rigid inclusion v^* is uniform.

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2. Isotropic Domain v. We assume that the elastic domain v with Young's modulus E and Poisson's ratio ν is isotropic. Then, the components of the tensor S in Eqs. (1.1) and (1.2) take the following form in the coordinate system fitted to the axes of symmetry of the ellipsoid v^* [2]:

$$S_{kkkk} = Qa_k^2 I_{kk} + RI_k, \qquad S_{kkll} = Qa_l^2 I_{kl} - RI_k,$$

$$2S_{klkl} = 2S_{kllk} = Q(a_k^2 + a_l^2)I_{kl} + R(I_k + I_l),$$

$$Q = 3/[8\pi(1-\nu)], \qquad R = (1-2\nu)/[8\pi(1-\nu)], \tag{2.1}$$

$$I_k = 2\pi a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{(a_k^2+u)\Delta}, \qquad I_{kk} = 2\pi a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{(a_k^2+u)^2\Delta},$$

$$3I_{kl} = 2\pi a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{(a_k^2+u)(a_l^2+u)\Delta}.$$

Here, a_k are the ellipsoid semi-axes, $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$ ($k, l = 1, 2, 3$; $k \neq l$; summation over k and l is not performed); the remaining components are $S_{klmn} = 0$.

The values of I_k, I_{kk}, and I_{kl} in Eq. (2.1) are expressed via elliptical integrals of the first and second kind and satisfy the relations [2]

$$I_1 + I_2 + I_3 = 4\pi, \quad I_{k1} + I_{k2} + I_{k3} = 4\pi/(3a_k^2), \quad a_1^2 I_{k1} + a_2^2 I_{k2} + a_3^2 I_{k3} = I_k,$$

$$I_{kl} = I_{lk} = (I_l - I_k)/[3(a_k^2 - a_l^2)] \qquad (k \neq l, a_k \neq a_l), \tag{2.2}$$

$$3I_{kl} = I_{kk} \qquad (k \neq l, a_k = a_l) \quad (k, l = 1, 2, 3),$$

which serve to find other indicated quantities based on the known values of I_1 and I_2. In particular, we have

$$I_1 = I_2 = I = 2\pi\delta(1-\delta^2)^{-3/2}[\arccos\delta - \delta(1-\delta^2)^{1/2}]; \tag{2.3}$$

$$I_3 = 4\pi - 2I, \qquad I_{11} = I_{22} = 3I_{12} = \frac{3I - 4\pi\delta^2}{4\alpha^2(1-\delta^2)},$$

$$I_{13} = I_{23} = \frac{4\pi - 3I}{3\alpha^2(1-\delta^2)}, \qquad I_{33} = \frac{4\pi(1-3\delta^2) + 6I\delta^2}{3\alpha^2\delta^2(1-\delta^2)}$$

for the oblate spheroid ($a_1 = a_2 = \alpha$, $a_3 = \delta\alpha$, and $\delta < 1$) and

$$I_2 = I_3 = I = 2\pi\delta^{-1}(\delta^{-2}-1)^{-3/2}[\delta^{-1}(\delta^{-2}-1)^{1/2} - \operatorname{arch}\delta^{-1}] \tag{2.4}$$

for the prolate spheroid ($a_1 = \alpha$, $a_2 = a_3 = \delta\alpha$, and $\delta < 1$). Then, Eq. (2.3) yields

$$I_1 = 4\pi - 2I, \qquad I_{11} = \frac{4\pi(3-\delta^2) - 6I}{3\alpha^2(1-\delta^2)},$$

$$I_{22} = I_{33} = 3I_{23} = \frac{4\pi - 3I\delta^2}{4\alpha^2\delta^2(1-\delta^2)}, \qquad I_{12} = I_{13} = \frac{3I - 4\pi}{3\alpha^2(1-\delta^2)}.$$

We present equality (1.2) in the matrix form as

$$s_{kl}\tilde{f}_l^* = f_k^\infty \qquad (k = 1, 2, \ldots, 6), \tag{2.5}$$

where $\tilde{f}_k^*$ and f_k^∞ are the components of six-dimensional vectors corresponding to the strain tensors $\tilde{\varepsilon}^*$ and ε^∞, s_{kl} are the elements of the 6×6 matrix of the form $s_{kl} = S_{kkll}$ ($k, l = 1, 2, 3$; summation over k and l is not performed); $s_{44} = 2S_{1212}$, $s_{55} = 2S_{1313}$, $s_{66} = 2S_{2323}$, and other elements s_{kl} are equal to zero.

From Eqs. (2.1), (2.5) and inequalities $s_{kk} > 0$ ($k = 4, 5, 6$; summation over k is not performed), it can be inferred that the above-formulated problem of determining the uniform stress field in an isolated ERI has a unique solution if the matrix $\|s^0_{kl}\| \equiv \|s_{kl}\|$ ($k, l = 1, 2, 3$) is nondegenerate:

$$\det \|s^0_{kl}\| \neq 0. \tag{2.6}$$

Let us consider some examples.

2.1. *ERI in the Form of an Oblate Spheroid.* Using Eqs. (2.1)–(2.3) and omitting cumbersome transformations, we find the value of the determinant $\Delta_0 \equiv \det \|s^0_{kl}\|$ indicated in Eq. (2.6):

$$\Delta_0 = \frac{1+\nu}{32\pi^2(1-\nu)^3(1-\delta^2)}\Big(\frac{3I - 4\pi\delta^2}{4(1-\delta^2)} + (1-2\nu)I\Big)$$

$$\times \Big[\Big(3 - 4\nu(1-\delta^2)\Big)I - 4\pi\delta^2 - (1-2\nu)(1-\delta^2)\,\frac{I^2}{\pi}\Big]. \tag{2.7}$$

It can be inferred from formulas (2.7), (2.3) that Δ_0 is a function of δ ($0 < \delta < 1$) and $3I > 4\pi\delta^2$. Substituting x for δ and introducing a new variable $t = 1 - x^2$ ($0 < t < 1$), we find that the condition $\Delta_0 = 0$ is equivalent to the equality

$$\Psi(t) \equiv 2(1-2\nu)tF^2 + (4\nu t - 3)F + 2(1-t) = 0, \tag{2.8}$$

where

$$F(t) \equiv I/(2\pi) = (1-t)^{1/2}t^{-3/2}[\arccos{(1-t)^{1/2}} - (1-t)^{1/2}t^{1/2}].$$

We consider the case of an incompressible elastic medium at $\nu = 0.5$. From expression (2.8), we find

$$f_1(t) \equiv \arccos{(1-t)^{1/2}} = f_2(t) \equiv 3(t - t^2)(3-2t)^{-1}. \tag{2.9}$$

As $f_1(0) = f_2(0) = 0$ and $f_1'(t) = 0.5(t-t^2)^{-1/2} > f_2'(t) = (4.5 - 6t)(3-2t)^{-2}(t-t^2)^{-1/2}$, Eq. (2.9) with respect to t at $0 < t < 1$ has no roots; therefore, $f_1(t) > f_2(t)$.

Thus, Eq. (2.8) at $\nu = 0.5$ has no solutions; therefore, $\Delta_0 \neq 0$ and the components of the stress tensor σ^* are unambiguously determined from Eq. (1.2).

Let $0 \le \nu < 0.5$. Considering (2.8) as an equation with respect to F, we find its roots

$$F_{1,2} = \frac{3 - 4\nu t \pm \sqrt{D}}{4(1-2\nu)t}, \qquad D = 16(1-\nu)^2t^2 - 8(2-\nu)t + 9 > 0, \tag{2.10}$$

where the inequality occurs because

$$\min_{0<t<1} D(t) = (1-2\nu)(5-4\nu)(1-\nu)^{-2} > 0.$$

For the function $\Psi(t)$, Eq. (2.8) yields

$$\Psi(t) = 2(1-2\nu)t(F - F_1)(F - F_2). \tag{2.11}$$

As $I_{kl} > 0$ ($k, l = 1, 2, 3$), it can be inferred from Eq. (2.3) that $(1-t)2/3 < F < 2/3$; from Eqs. (2.10) and $9 - 8t + 4\nu t > 0$, we find the inequality

$$F_1 > \frac{3 - 4\nu t}{4(1-2\nu)t} > \frac{2}{3}.$$

Therefore, $F - F_1 < 0$ and the condition $\Psi(t) = 0$ by virtue of Eq. (2.11) is equivalent to $F = F_2$. Let us show that this equation with respect to t has no roots at $0 < t < 1$.

In fact, as $\nu \to 0.5$, the inequality $f_1(t) > f_2(t)$ for the functions from (2.9) is equivalent to the inequality

$$F > F_2 \equiv \frac{3 - 4\nu t - \sqrt{D}}{4(1-2\nu)t} = \frac{4(1-t)}{3 - 4\nu t + \sqrt{D}} \tag{2.12}$$

[F, F_2, and D are determined in Eqs. (2.8) and (2.10)].

Considering F_2 as a function of ν, we find its derivative

$$F_2'(\nu) = \frac{16(1-t)t}{(3 - 4\nu t + \sqrt{D})^2}\Big(1 + \frac{4(1-\nu)t - 1}{\sqrt{D}}\Big).$$

It can be inferred that $F_2'(\nu) > 0$ at $f_3 \equiv 4(1-\nu)t - 1 \geq 0$. At $f_3 < 0$, we also have $F_2'(\nu) > 0$, because $D-(-f_3)^2 = 8(1-t) > 0$. Therefore, the function $F_2 = F_2(\nu)$ is increasing, and we obtain $F > F_2\Big|_{\nu=0.5} > F_2\Big|_{\nu<0.5}$ from Eq. (2.12). Then, considering that $F - F_1 < 0$, we find from Eqs. (2.10) and (2.11) that

$$\Psi(t) < 0 \quad \text{at} \quad 0 < t < 1 \quad \text{and} \quad 0 \leq \nu \leq 0.5.$$

2.2. *ERI in the Form of a Prolate Spheroid.* For an ERI taking the form of a prolate spheroid, we performed calculations similar to those in Sec. 2.1 and found from Eqs. (2.1), (2.2), and (2.4) that the condition $\det \|s_{kl}^0\| = 0$ is equivalent to the equality

$$\Psi^0(t) \equiv 2(1-2\nu)t(F^0)^2 + [(4\nu-3)t+3]F^0 - 2 = 0, \tag{2.13}$$

where

$$F^0 \equiv \frac{1}{t} - \frac{1-t}{t^{3/2}} \ln \frac{1+t^{1/2}}{(1-t)^{1/2}}.$$

At $\nu = 0.5$, we have

$$f_1^0(t) \equiv \ln \frac{1+t^{1/2}}{(1-t)^{1/2}} = f_2^0(t) \equiv \frac{3t^{1/2}}{3-t}. \tag{2.14}$$

Similar to relation (2.9), Eq. (2.14) at $0 < t < 1$ has no roots because $f_1^0(0) = f_2^0(0) = 0$ and

$$(f_1^0(t))' = \frac{1+t^{1/2}}{2(1+t^{1/2})(1-t)} > (f_2^0(t))' = \frac{3}{2}\,\frac{3t^{-1/2}+t^{1/2}}{(3-t)^2}.$$

Therefore, $f_1^0(t) > f_2^0(t)$.

Similar to Eq. (2.11), Eq. (2.13) at $0 \leq \nu < 0.5$ yields

$$\Psi^0(t) = 2(1-2\nu)t(F^0 - F_1^0)(F^0 - F_2^0),$$

$$F_1^0 \equiv \frac{(3-4\nu)t - 3 + \sqrt{D_0}}{4(1-2\nu)t} = \frac{4}{3-(3-4\nu)t+\sqrt{D_0}}, \tag{2.15}$$

$$F_2^0 = \frac{(3-4\nu)t - 3 - \sqrt{D_0}}{4(1-2\nu)t} < 0, \qquad D_0 = (3-4\nu)^2t^2 - 2(1+4\nu)t + 9 > 0.$$

The condition $\Psi^0(t) = 0$ reduces to the equation $F^0 = F_1^0$, which has no solutions at $0 < t < 1$. Indeed, for the functions from (2.14), the inequality $f_1^0(t) > f_2^0(t)$ is equivalent to $F^0 < F_1^0$ at $\nu \to 0.5$. As $[1+(3-4\nu)t]^2 - D_0 = 8(t-1) < 0$, the derivative of the function F_1^0 with respect to ν is

$$(F_1^0(\nu))' = \frac{16t}{[3-(3-4\nu)t+\sqrt{D_0}\,]^2}\Big(\frac{1+(3-4\nu)t}{\sqrt{D_0}} - 1\Big) < 0.$$

Therefore, the function $F_1^0 = F_1^0(\nu)$ is decreasing, and

$$F^0 < F_1^0\Big|_{\nu=0.5} < F_1^0\Big|_{\nu<0.5},$$

while Eqs. (2.15) yield

$$\Psi^0(t) < 0 \quad \text{at} \quad 0 < t < 1 \quad \text{and} \quad 0 \leq \nu \leq 0.5.$$

Thus, it is shown that condition (2.6) is satisfied for the considered isolated rigid inclusions in the form of oblate and prolate spheroids. This fact provides unique solutions of problems with determining stresses in these inclusions under external forces uniformly distributed at infinity.

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